9.3: Separable Differential Equations

Entry Task: *(Motivation)*

Implicitly differentiate $x^2 + y^3 = 8$ and solve for $\frac{dy}{dx}$.

$$\frac{d}{dx}\left[x^2 + y^3 = 8\right]$$

$$\Rightarrow \quad 2x + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 3y^2\frac{dy}{dx} = -2x$$

$$\Rightarrow \quad \boxed{\frac{dy}{dx} = \frac{-2x}{3y^2}}$$

Idea: separate and integrate both sides.

Entry Task continued:

Find the *explicit* solution for $\frac{dy}{dx} = \frac{-2x}{3y^2}$ with $y(0) = 2$.

$$\frac{dy}{dx} = -\frac{2x}{3y^2}$$

$$\Rightarrow \quad 3y^2\frac{dy}{dx} = -2x$$

$$\int 3y^2\,dy = \int -2x\,dx$$

$$y^3 + C_1 = -x^2 + C_2$$

$$\Rightarrow \quad x^2 + y^3 = \underbrace{C_2 - C_1}_{\leftarrow \text{ A constant}}$$

$$x^2 + y^3 = C$$

$$y(0) = 2 \Rightarrow 0^2 + 2^3 = C \Rightarrow C = 8$$

$$x^2 + y^3 = 8$$

$$\Rightarrow \quad y^3 = 8 - x^2$$

$$\Rightarrow \quad \boxed{y = (8 - x^2)^{1/3} \leftarrow \text{EXPLICIT}}$$

9.3: Separable Differential Equations

A **separable** differential equation is one that can be written as:

$$\frac{dy}{dx} = f(x)g(y).$$

(or $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ or $\frac{dy}{dx} = \frac{g(y)}{f(x)}$.)

Example: Find the explicit solution to $\frac{dy}{dx} = \frac{x}{y^4}$ with $y(0) = 1$.

$$y^4 \frac{dy}{dx} = x$$

$$\int y^4\,dy = \int x\,dx$$

$$\frac{1}{5}y^5 = \frac{1}{2}x^2 + C_1$$

$$y^5 = \frac{5}{2}x^2 + 5C_1$$

Let $C_2 = 5C_1$

$$y^5 = \frac{5}{2}x^2 + C_2$$

$$y = \left(\frac{5}{2}x^2 + C_2\right)^{1/5}$$

GENERAL SOL'N

$$\boxed{y = \left(\frac{5}{2}x^2 + C\right)^{1/5}}$$

$y(0) = 1$

$$\Rightarrow 1 = \left(\frac{5}{2}(0)^2 + C\right)^{1/5}$$

$$\Rightarrow 1^5 = C \Rightarrow \boxed{C = 1}$$

$$y = \left(\frac{5}{2}x^2 + 1\right)^{1/5}$$

Q) What is C_1?
What is C_2?

$$\frac{1}{5}(1)^5 = \frac{1}{2}(0) + C_1 \Rightarrow \boxed{C_1 = 1/5}$$

$$\boxed{C_2 = 5C_1 = 1!}$$

Example: Find the explicit solution to
$\frac{dy}{dx} = \frac{x\sin(2x)}{3y}$ with $y(0) = -1$.

$u = x \quad dv = \sin(2x)\,dx$

$du = dx \quad v = -\frac{1}{2}\cos(2x)$

$$\int 3y\,dy = \int x\sin(2x)\,dx$$

$$\frac{3}{2}y^2 = -\frac{1}{2}x\cos(2x) - \int -\frac{1}{2}\cos(2x)\,dx$$

$$\frac{3}{2}y^2 = -\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x) + C_1$$

$C_2 = \frac{2}{3}C_1$

$$y^2 = -\frac{1}{3}x\cos(2x) + \frac{1}{6}\sin(2x) + C_2$$

$$\boxed{y = \pm\sqrt{-\frac{1}{3}x\cos(2x) + \frac{1}{6}\sin(2x) + C}}$$

GENERAL EXPLICIT SOL'N.

$y(0) = -1 \Rightarrow$ ① THIS IS "$-$"

$\Rightarrow$ ② $-1 = -\sqrt{0 + 0 + C}$

$\Rightarrow \boxed{C = 1}$

Q) what is C_1?

$C_1 = \frac{3}{2}$

$$y = -\sqrt{-\frac{1}{3}x\cos(2x) + \frac{1}{6}\sin(2x) + 1}$$

Example: Find the explicit solution to

$$(x+1)\frac{dy}{dx} = \frac{x^2}{e^y} \quad \text{with } y(0) = 0.$$

$$e^y \frac{dy}{dx} = \frac{x^2}{x+1}$$

$$\begin{array}{r} x - 1 \\ x+1 \overline{\big) \; x^2 \qquad} \\ -(x^2 + x) \\ \hline -x \\ -x - 1 \\ \hline 1 \end{array}$$

$$\int e^y\, dy = \int \frac{x^2}{x+1}\, dx$$

$$\int e^y dy = \int x - 1 + \frac{1}{x+1}\, dx$$

$$e^y = \frac{1}{2}x^2 - x + \ln|x+1| + C_1$$

$$\boxed{y = \ln\left(\frac{1}{2}x^2 - x + \ln|x+1| + C\right)}$$ general sol'n

$y(0) = 0 \Rightarrow 0 = \ln(0 - 0 + 0 + c)$

$\Rightarrow \boxed{C = 1}$

$$\boxed{y = \ln\left(\frac{1}{2}x^2 - x + \ln|x+1| + 1\right)}$$

Law of Natural Growth

Assumption: *"The rate of growth of a population is proportional to the size of the population."*

$P(t)$ = population at year *t*,

$\frac{dP}{dt} =$ rate of change of the population

The law of natural growth assumes

$$\frac{dP}{dt} = kP,$$

for some constant *k*

(we call *k* the *relative growth rate*).

Find the explicit solution to

$$\frac{dP}{dt} = kP \text{ with } P(0) = P_0$$

$$\boxed{P(t) = P_0 e^{kt}}$$

$$\int \frac{1}{P}\,dP = \int k\,dt$$

$$\ln|P| = kt + C_1$$

$$\Rightarrow |P| = e^{(kt + C_1)}$$

$$\Rightarrow |P| = e^{C_1} e^{kt}$$

Let $C_2 = e^{C_1}$

$$|P| = C_2 e^{kt}$$

$$\Rightarrow P = \pm C_2 e^{kt}$$

Let $C_3 = \pm C_2$

$$P(t) = C_3 e^{kt}$$

$$\boxed{\text{GENERAL SOL'N } P(t) = C e^{kt}}$$

CLEANER THAN WRITING

$$\underbrace{P(t) = \pm e^{C_1} e^{kt}}_{\text{also correct}}$$

$$P(0) = P_0 \Rightarrow P_0 = C e^0 \Rightarrow \boxed{C = P_0}$$

WHAT IS C_1?

$$\pm e^{C_1} = P_0$$

$$\Rightarrow C_1 = \ln|P_0|$$

1. 500 bacteria are in a dish at t=0hr.
8000 bacteria are in the dish at t=3hr.
Assume the population grows at a rate proportional to its size.
Find the function, B(t), for the bacteria population with respect to time.

$$\frac{dB}{dt} = kB \qquad B(0) = 500$$

USING THE GENERAL SOL'N WE ALREADY FOUND

$$B(t) = B_0 e^{kt}$$

- $B(0) = 500 \Rightarrow B_0 = 500$

$$B(t) = 500e^{kt}$$

- $B(3) = 8000 \Rightarrow$

$$500e^{3k} = 8000$$

$$\Rightarrow e^{3k} = 16$$

$$3k = \ln(16) \Rightarrow k = \frac{\ln(16)}{3} \simeq 0.924196$$

ROUGHLY 92% growth per hour

THUS,

$$\boxed{B(t) = 500e^{\frac{\ln(16)}{3}t}}$$

$$B(t) = 500(16)^{t/3}$$

NOTE:

$$\frac{t}{3}\ln(16) = \ln(16^{t/3})$$

$$\Rightarrow e^{\frac{t}{3}\ln(16)} = e^{\ln(16^{t/3})} = 16^{t/3}$$

2. The *half-life* of cesium-137 is 30 years. Suppose we start with a 100-mg sample. The mass decays at a rate proportional to its size. Find the function, m(t), for the mass with respect to time.

$$\frac{dm}{dt} = km, \quad m(0) = 100$$

$$m(t) = m_0 e^{kt}$$

- $m(0) = 100 \Rightarrow m_0 = 100$

$$m(t) = 100 e^{kt}$$

- $m(30) = 50 \leftarrow$ HALF!

$$50 = 100 e^{30k}$$

$$\Rightarrow \frac{1}{2} = e^{30k}$$

$$\Rightarrow \ln(\tfrac{1}{2}) = 30k$$

$$\Rightarrow k = \frac{1}{30}\ln(\tfrac{1}{2})$$

$$\approx -0.023104$$

NEGATIVE! DECAY!

lose about 2% per year

$$\boxed{m(t) = 100 e^{\frac{\ln(1/2)}{30}t}}$$

$$= 100 e^{\ln((\frac{1}{2})^{t/30})}$$

$$= 100 \left(\tfrac{1}{2}\right)^{t/30}$$

3. You invest \$10,000 into a savings account and never make any deposits or withdrawals. The balance grows at a rate proportional to its size (i.e. *interest* is a percentage of the balance at any time). In 3 years, you notice your balance is \$10,400.
Find the function, A(t), for the amount of money in the account with respect to time.

ANNUAL INTEREST RATE COMPOUNDED CONTINUOUSLY

$$\frac{dA}{dt} = kA$$

≈ AMOUNT OF INTEREST ADDED TO THE ACCOUNT PER YEAR

$$A(t) = Ce^{kt}$$

$$A(0) = 10000 \Rightarrow 10000 = Ce^{0} = C$$

$$A(3) = 10400 \Rightarrow 10400 = 10000e^{k(3)}$$

$$\Rightarrow 1.04 = e^{k(3)}$$

$$\ln(1.04) = 3k$$

$$k = \frac{1}{3}\ln(1.04) \approx 0.0130736$$

1.31% annual interest compounded continuously

$$A(t) = 10\,000\,e^{\frac{1}{3}\ln(1.04)t}$$

$$\approx 10000\,e^{0.013073t}$$